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EXISTENTIALLY-MUTE THEORIES AND EXISTENCE UNDER ASSUMPTIONS

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 02D99, 02E05

<u>Existentially-mute theories and existence under assumptions.</u>

0.1 <u>Synopsis.</u> The notion of $\exists$-mute theories is a generalization of Kleene's [62] stroke relation, i.e. - the single - axiom theory {C} (where C is a closed fla) is $\exists$-mute exactly when $C|C$. We show that the property

$$T \vdash \exists x Ax \implies \exists n\ T \vdash A\bar{n}$$

(provability in int. arith.; $\exists xAx$ closed, n a numeral) holds exactly when T is $\exists$-mute.

As a simple application of the method of proof we obtain the result (already proved by Smorynski [71]) that classical arithmetic is not an extension of bounded complexity of intuitionistic arithmetic.

Next we relate the notion of $\exists$-mute theories to Kleene's stroke relation, and we exhibit some simple structural properties of $\exists$-mute theories.

Finally we deal with theories which may replace intuitionistic arithmetic in the treatment above. We call these "$\exists$-stable theories". We show that any extension of int. arith. with an "$\exists$-valid" rule is $\exists$-stable, where a rule $\frac{\{A_i\}_i}{B}$ is $\exists$-valid if for every $\exists$-mute theory T

$$\{T \vdash A_i\}_i \implies T \vdash B \ .$$

Among the $\exists$-valid rules we find the uniform reflection principle, but not Markov scheme.

Our main results are only a slight generalization of theorems of Scarpellini [72]. New are only the presentation in a system of natural deduction, and some peripherial material.

0.2. <u>Preliminary conventions and notations.</u> We use throughout Gentzen's natural deduction system for intuitionistic arithmetic, and the notions and results of the metatheory developped by Prawitz ([65],[70]) for it.

We restrict our attention to the disjunction-free part of intuitionistic arithmetic, since the $\exists$-quantifier is intuitionistically definable in this fragment (for details vid. Leivant [73]). But we still write A v B as a shorthand for

$$\exists x\{[x = 0 \to A] \ \& \ [x = 1 \to B]\} \ .$$

We use α, α_0, β etc. as meta-parameters for sets of **flas**.

ϕ, ψ etc. are meta-parameters for functionals, i.e. - functions whose domain and range will be defined every time explicitly. We say that ϕ is p.r. when the arithmetical function

$$\phi^* : \ulcorner x \urcorner \to \ulcorner \phi x \urcorner \qquad \text{is p.r. ,}$$

$\ulcorner x \urcorner$ being the Gödel-number of the object x.

S_A^{++} is the set of strictly-positive subformulas of the formula A, and

$$S_\alpha^{++} = \bigcup_{A \in \alpha} S_A^{++} \ .$$

If n is a natural number, then $\bar{n}$ denotes the numeral n, i.e. -

$$\bar{0} = 0 \ ; \qquad \overline{n+1} = \bar{n}' \ .$$

$\underline{A}$, $\underline{B}$ etc. denote occurrences of the flas A, B... .

IA denotes intuitionistic (Heyting's) arithmetic,

CA - classical (Peano's) arithmetic.

1. Existentially-mute theories.

1.1. Definitions.

Let α be a set of closed (and disjunction free) flas of (first-order) arithmetic, and let ϕ be a functioned from derivations to natural numbers.

We define a set of flas $S_\alpha^\phi \subseteq S_\alpha^{++}$ as follows:

(i) $A \in \alpha \Longrightarrow A \in S_\alpha^\phi$

(ij) $A \ \& \ B \in S_\alpha^\phi \Longrightarrow A, B \in S_\alpha^\phi$

(iij) $A \to B \in S_\alpha^\phi \Longrightarrow B \in S_\alpha^\phi$

(iv) $\forall xAx \in S^{\phi}_{\alpha} \Longrightarrow A\bar{n} \in S^{\phi}_{\alpha}$ for every natural number n.
(v) $\exists xAx \in S^{\phi}_{\alpha}$ and Π is a derivation for $\alpha \vdash \exists xAx$, then $A(\phi\Pi) \in S^{\phi}_{\alpha}$.
(vi) Only flas which can be shown to be in S^{ϕ}_{α} by clauses (i)-(v) are in S^{ϕ}_{α}.

Let ψ be a functional from derivations to derivations. We say that ψ is a confirmation functioned of ϕ/α, if whenever

$$\exists xAx \in S^{\phi}_{\alpha} \quad \text{and } \Pi \text{ is a derivation for } \alpha \vdash \exists xAx,$$

then $\psi\Pi$ is a derivation for $\alpha \vdash A(\phi\Pi)$. ϕ is a choice-functional for α if there is a confirmation functional of ϕ/α.

If there is a choice functional for α we say that α is $\exists$-mute, and if not - that α is essentially-existential.

1.2. Theorem: Let α be $\exists$-mute, $\exists xAx$ closed.

$$\alpha \vdash \exists xAx \Longrightarrow \exists n \; \alpha \vdash A\bar{n} \; .$$

More precisely: Let ϕ be a choice-functional for α, and ψ a confirmation functional for ϕ. If Π is a derivation for $\alpha \vdash \exists xAx$ then we can find, primitive-recursively in ϕ, a number n, and primitive-recursively in ψ - a derivation Π^{*} for $\alpha \vdash A\bar{n}$.

The proof occupies the rest of this section.

1.2.1 Let us say that an occurrence $\underline{\exists xF}$ in a derivation Π is α-critical if
(i) $\exists xF$ is a closed fla.
(ij) $\underline{\exists xF}$ is the major premise of an instance of $\exists E$.
(iij) $\underline{\exists xF}$ depends in Π only on flas in α.

1.2.2. Lemma: Let α,ϕ,ψ be as in the theorem. Every derivation Π for $\alpha \vdash A$ is transformable (by instructions that are p.r. in ψ) to a derivation Π° of the form

$$\Pi^{\circ} \equiv \begin{matrix} \Sigma_i \\ \{Fi\} \; i < n \\ \Pi^{1} \end{matrix} \qquad \text{for } \alpha \vdash A$$

s.t. (i)Π^1 is normal and without redundant parameters (n.w.r.p. for short);

(ij) there is no α-critical occurrence in Π^1;

(iij) $F_i \in S^{\phi}_{\alpha}$
(iv) $\sum_i \in$ range ψ } $i < n$.

Proof of the lemma: by induction on the complexity of derivations, i.e. - by cases of inference-rules. The only non-trivial case is an instance of $\exists E$ where the major premise is α-critical, i.e. -

$$\sum \equiv \quad \frac{\begin{matrix} \Delta_i \\ \alpha\ \{G_i\} \\ \Delta^1 \\ \exists x A x \end{matrix} \qquad \begin{matrix} \Gamma_j \\ \alpha\ \{H_j\}\ [Aa] \\ \Gamma^1(a) \\ B \end{matrix}}{B}\ \exists E$$

with conditions (i)-(iv) on the main subderivations, by ind. ass..

The part $\begin{matrix} \alpha\ \{G_i\} \\ \Delta^1 \\ \exists x A x \end{matrix}$ of $\sum$

being normal, every main branch σ of Δ^1 has only an elimination-part. $\exists xAx$ is critical, therefore closed, so $\exists xAx$ is a strictly-positive subformula of the top-formula C of σ, which belongs to $\alpha \cup \{G_i\} \subseteq S^{\phi}_{\alpha}$.

Furthermore, no existential fla other than $\exists xAx$ may occur in σ, because such an occurrence would necessarily be closed, and would be the major premise of an instance of $\exists E$, and hence α-critical, contradicting the ind. ass.. So $\exists xAx$ is not a s.p. subformula of any s.p. existential subformula of C, and therefore $\exists xAx \in S^{\phi}_{\alpha}$ (disregarding the definition of ϕ).

Define now

$$\sum^1 \equiv \text{Df} \quad \begin{matrix} \Gamma_j & \psi\Delta \\ \alpha\ \{H_j\} & [A(\phi\Delta)] \\ & \Gamma^1(\phi\Delta) \\ & B \end{matrix}$$

which obviously possess the desired properties.

1.2.3 Proof of the theorem.

Let the Π of the theorem be given, and let

$$\Pi^{\circ} \equiv \begin{array}{c} \Sigma_i \\ \{F_i\}\ \alpha \\ \Pi^1 \\ \hline \exists xAx \end{array}$$

be obtained from Π by the lemma.

Case 1: The last inference of Π° is an introduction, $\frac{At}{\exists xAx}$ say. Since Π° is n.w.r.p. t must be free of parameters, and therefore it represents a numeral. Take

$$\Pi^{*} \equiv \begin{array}{c} \Sigma_i \\ \{F_i\}\ \alpha \\ \Pi^1 \end{array} .$$

Case 2: The last inference of Π° is an elimination. Repeat the main argument in the proof of the lemma to conclude that $\exists xAx \in S_{\alpha}^{\phi}$, and take

$$\Pi^{*} \equiv \psi\Pi^{\circ} .$$

Case 3: The last inference is an instance of the λ-rule - trivial.
In any case, Π^{*} is a derivation for $\alpha \vdash A\bar{n}$ for some n, and obviously satisfies the desired properties.

1.3. Examples of $\exists$-mute theories.

1.3.1 Assume that for every $\exists xAx \in S_{\alpha}^{\#}$ we have

$$(*)\ \alpha \vdash \exists xAx \implies \alpha \vdash A\bar{n} \text{ for some n.}$$

Then α is trivially $\exists$-mute. Theorem 1.2. was proved for this case, which is certainly the most useful, by Scarpellini [72].

Note, however, that in general we have here redundant information about α, and that S_α^ϕ is a smaller set than S_α^{++}. If $\exists xAx \in S_\alpha^\phi$, then $An \in S_\alpha^{++}$ for every n and in any case, while $An \in S_\alpha^\phi$ only if there is a derivation Π for $\alpha \vdash \exists xAx$, and only for $n=\phi\Pi$. After we know th. 1.2, (*) holds for every $\exists$-mute α, of course.

1.3.2. The set of all classically-true flas is $\exists$-mute. The set of classically provable flas is not.

1.3.3. Every theory α s.t. S_α^{++} is free of disjunction and $\exists$ is $\exists$-mute. E.g.- extend int. arith. with the scheme

$$\forall x \neg\neg A \rightarrow \neg\neg\forall xA \ .$$

1.3.4. If every $A \in \alpha$ is classically true, and for every $\exists xAx \in S_\alpha^{++}$ and every n An is decidable, then α is $\exists$-mute. For then, if $\alpha \vdash \exists xAx$ for $\exists xAx \in S_\alpha^{++}$ then $\exists xAx$ is true, hence for some n $A\bar{n}$ is true. $A\bar{n}$ is decidable, therefore $\vdash A\bar{n}$.

This is in particular the case if for every $\exists xA \in S_\alpha^{++}$ A is quantifier-free, e.g. - if α consists of almost-negative classically-true formulas. α may thus be taken to be the set of all instances of the (p.r.) Markov scheme

$$\neg\forall xAx \rightarrow \exists x \neg Ax \qquad (A \text{ q.f.}).$$

Also, α may contain true assertions of realizability, i.e. - flas of the form n $\underset{\sim}{r}$ A where n classically-realizes A, because these flas are equivalent in intuitionistic arithmetic to almost negative flas (vid. Troelstra [70] 3.7).

1.3.5. Remark: Note that the condition $A \in \alpha \implies$ A classically true in example 1.3.4. may not be dropped. Take as an example $\alpha = \{\neg G, M\}$

where

$$G \equiv \forall x \neg \, Prov(x, \ulcorner G \urcorner) \qquad \text{(Gödel's fla)}$$

and M is an instance of Markov scheme:

$$M \equiv \neg\forall x \neg Prov(x, \ulcorner G \urcorner) \rightarrow \exists x \neg\neg Prov(x, \ulcorner G \urcorner).$$

M is classically true, $\neg G$ is not, and both are almost-negative.
α is not $\exists$-mute:

$$\alpha \vdash \exists x \neg\neg Prov(x, \ulcorner G \urcorner)$$

hence, $Prov(x, \ulcorner G \urcorner)$ being p.r.,

$$\alpha \vdash \exists x Prov(x, \ulcorner G \urcorner) \ .$$

If α is $\exists$-mute, then

$$\alpha \vdash Prov(n, \ulcorner G \urcorner) \text{ for some } n.$$

But G is unprovable, so

$$\alpha \vdash 0{=}1$$

or $\qquad M \vdash \neg\neg G$

and hence, classically, $\vdash G$,contradicting the famous underivability of G.

2. A simple application of the method:
Classical arithmetic is not an extension of bounded quantifier-complexity of intuitionistic arithmetic.

As a corollary of the hierarchy theorem (see Rogers [67] §14.7.X), and also as a particular case of Kreisel-Lévy [68] th.4, we have

2.1 Lemma: Given n, there is a close formula F_n which is undecidable from true formulas of qu.-complexity <n.

2.2. Theorem (Smorynski [71]): Classical arith. (CA) is not an extension of bounded complexity of intuitionistic arith. (IA).

Proof: Suppose τ is a set of true flas of qu.-complexity <n, s.t.(*) $CA \supseteq IA + \tau$. Let F_n be given by the lemma. By (*), there is a derivation Δ (in Gentzen's system of natural deduction for IA), which we may assume to be normal and without redundant parameters (in the sense of Prawitz [70]), for

$$\tau \vdash_{IA} F_n \vee \neg F_n \ .$$

(1) If the last inference in Δ is an instance of the $\bigwedge$-rule, then we get trivially $\tau \vdash F_n$, contradicting the choice of F_n.
(2) If the last inference is an introduction, then τ decides F_n, contradiction again.
(3) The last inference is not an instance of induction, because Δ is normal, and F_n close.
(4) If the last inference is an elimination, let σ be a main path in Δ. σ is then composed of elimination-steps only, and $F_n \vee \neg F_n$ is a subformula of the top-formula A of σ. Necessarily then, the qu.-comp. of $A \geq n$.

A cannot be the conclusion of an instance of induction, because Δ is n.w.r.p. So $A \in \tau$, contradicting the assumption that τ is of qu. com. <n.

Thus we get a contradiction from (*).

2.3. Remark: Smorynski has proved a more general result: he showed that

$$CA \not\supseteq IA + \tau$$

for every consistent τ of bounded complexity (not necessarily composed of true formulas only).

3. Connection with Kleene's stroke-relation.

3.1. Let A be a closed fla. By th. 1.2 {A} is $\exists$-mute iff for every closed $\exists xCx$

$$A \vdash \exists xCx \Longrightarrow \exists n\ A \vdash C\bar{n}\ .$$

So, by Kleene [62] (2.2., 2.7., 2.11) {A} is $\exists$-mute iff $A|A$.

This may be proved directly, as a corollary of the following

Proposition: Let ϕ be a choice-functional for {A}, where A is a closed fla. For every $B \in S^{\phi}_{A}$, $A \vdash B \Longrightarrow A|B$.

Proof: By induction on the complexity of B.

(1) Basis: The "smallest" flas in S^{ϕ}_{A} (i.e. – the flas $\underline{F} \in S^{A}_{\phi}$ s.t. no proper subformula of $\underline{F}$ is in S^{A}_{ϕ}) are, by the definition of S^{ϕ}_{A}, either (i) prime flas; or (ij) existential flas $\exists xGx$ s.t. $A \nvdash \exists xGx$.
In the first case the lemma holds trivially, and in the second – by the definition of $|$.

(2) $B \,\&\, C \in S^{\phi}_{A}$, $\quad A \vdash B \,\&\, C \Longrightarrow$
$B, C \in S^{\phi}_{A}$, $\quad A \vdash B\ ,\ A \vdash C \Longrightarrow$ (ind. hyp.)
$A \mid B\ ,\ A \mid C \Longrightarrow A \mid B\&C$.

(3) $(B \to C) \in S^{\phi}_{A} \Longrightarrow C \in S^{\phi}_{A}$
If $A \vdash B \to C$ and $A \vdash B$ then $A \vdash C$; hence, by ind. hyp. $A \mid C$. So $A \vdash B \Longrightarrow A \mid C$, i.e. – $A \mid B \to C$.
(Note that the condition $A \mid B$ in the definition of $A \mid B \to C$ is not used here).

(4) $\forall xBx \in S^{\phi}_{A} \Longrightarrow \forall n\ Bn \in S^{\phi}_{A}$.
So, $A \vdash \forall xBx \Longrightarrow \forall n\ A \vdash Bn$

$\Longrightarrow \forall n\ A \mid Bn$ (ind. hyp)

$\Longrightarrow A \mid \forall xBx$.

(5) $\exists xBx \in S^{\phi}_{A}$, $A \vdash \exists xBx$ by a derivation Π

$\Longrightarrow A \vdash B(\phi\Pi)$, $B(\phi\Pi) \in S^{\phi}_{A}$.

So, by ind. hyp. $A \mid\!\vdash B(\phi\Pi)$, and $A \mid \exists xBx$.

3.2. The equivalent 3.1 has the following peculiarity: while the definition of $A \mid A$ involves all subformulas of A, the definition of "{A} is $\exists$-mute" involves only (part of) S^{++}_{A}. The following proposition may throw some light on this.

<u>Proposition</u>: If $C \to \exists xBx \vdash \exists xBx$ (where $C \to \exists xBx$ is closed), then either (i) $C \to \exists xBx \vdash C$ (and hence $\vdash \neg\neg C$);
or (ij) $C \to \exists xBx \vdash B\bar{n}$ for some n.

The proof is very similar to that of th. 2.

4. Structural properties of $\exists$-mute theories.

4.0. If α is a finite set (of closed flas), then α is $\exists$-mute iff $(\wedge\alpha) \mid (\wedge\alpha)$, where $(\wedge\alpha)$ is the conjunction of all flas in α. So, the main interest in $\exists$-mute theories concerns those which are not finitely axiomatizable, and to which the method of Kleene's $\mid$ does not apply. There is therefore some interest in considering the relation between a set of flas and its finite subsets, w.r.t. the notion of $\exists$-muteness.

4.1. We have, first, the obvious "compactness" property:
<u>Proposition</u>: If every finite $\alpha_0 \subset \alpha$ is $\exists$-mute, then α is $\exists$-mute. We have, more generally: if for every finite $\alpha_0 \subset \alpha$ there is an $\exists$-mute α'_0 s.t. $\alpha_0 \subseteq \alpha'_0 \subseteq \alpha$, then α is $\exists$-mute.
Proof: If $\alpha \vdash \exists xCx$ ($\exists xCx$ closed) then for some finite α_0, $\alpha_0 \vdash \exists xCx$, hence $\alpha'_0 \vdash C\bar{n}$ and $\alpha \vdash C\bar{n}$ for some n.

4.2. For essentially-existential (i.e. - not $\exists$-mute) theories this compactness properly may fail. That is,

Proposition: There is an $\exists$-mute theory α s.t. every finite $\alpha_0 \subset \alpha$ is essentially existential.

Proof: Arithmetic is essentially undecidable, so there is a sequence of true flas $\{A_n\}_{n<\omega}$ s.t. A_n is not decided in $\mathcal{CA}$ by $\bigwedge_{i<n} A_i$. Let us write every A_n in the $\{\forall, \&, \bigwedge\}$ - fragment.

Define $\quad B_0 \equiv_{Df} A_1 \vee \neg A_1$

and for $\quad n > 1 \quad B_n \equiv_{Df} A_n \& [A_{n+1} \vee \neg A_{n+1}]$

$$\mathcal{B} =_{Df} \{B_k\}_{k<\omega} .$$

Classically $\quad \vdash_{\mathcal{CA}} B_k \to A_k . \qquad (k=1,2,\dots)$

$\mathcal{B}$ is $\exists$-mute trivially, but no finite $\mathcal{B}_0 \subset \mathcal{B}$ is $\exists$-mute: Suppose that some finite $\mathcal{B}_0 = \{B_{i_j}\}_{j<n} \subset \mathcal{B}$ is $\exists$-mute. Let $m = \max_{j<n} \{i_j\}$.

We have then $\mathcal{B}_0 \vdash_{\mathcal{IA}} A_m \vee \neg A_m$, and $\mathcal{B}_0$ being $\exists$-mute, $\mathcal{B}_0 \vdash_{\mathcal{IA}} A_m$. ($\neg A_m$ is false). But then $\bigwedge_{i<m} A_i \vdash_{\mathcal{CA}} A_m$, contradicting the construction of $\{A_n\}_{n<\omega}$

4.3. Let us finally note that the "compactness" properly in 4.1 cannot be strengthen by fixing a certain bound on the size of the finite $\alpha_0 \subset \alpha$ which have to be $\exists$-mute. In other words:

Proposition: For every natural number n, there is an essentially-existential set α s.t. every $\alpha_0 \subset \alpha$ with less than n formulas in $\exists$-mute.

4.3.1. Lemma: There is a sequence

$$\mathcal{A} = \{A_i\}_{i<\omega} \text{ of true flas, s.t. for no n,}$$

$$\mathcal{A} \sim \{A_n\} \vdash_{\mathcal{CA}} A_n .$$

<u>Proof</u>: Let A_0 be undecidable. Assume that $\{A_i\}_{i<n}$ is defined, and let B_n be some true $\{A_i\}_{i<n}$ - undecidable fla. Define

$$A_n \equiv \bigwedge_{i<n} A_i \to B_n .$$

We show now that $A \sim \{A_n\} \nvdash_{CA} A_n$. It is enough to see that if

(*) $\{A_i\}_{i<k} \sim \{A_n\} \nvdash_{CA} A_n$, then

(**) $\{A_i\}_{i\leq k} \sim \{A_n\} \nvdash_{CA} A_n$,

because then A_n is unprovable from any finite subset of $A \sim \{A_n\}$. Suppose (*), and assume first that k > n, and that not (**), i.e. -

(***) $\{A_i\}_{\substack{i\leq k \\ i\neq n}} \vdash_{CA} A_n$

$A_k \equiv \bigwedge_{i<k} A_i \to B_k$, so, since k > n,

$A_n \to B_k \vdash A_k$, and from (***)

then $\{A_i\}_{\substack{i<k \\ i\neq n}} , A_n \to B_k \vdash A_n$.

From classical propositional calculus then

$$\{A_i\}_{\substack{i<k \\ i\neq n}} \vdash A_n ,$$

Contradicting (*).
Assume now k < n, i.e. -

$$\{A_i\}_{i\leq k} \vdash A_n$$

then $\{A_i\}_{i<n} \vdash A_n$

i.e. - $\{A_i\}_{i<n} \vdash \bigwedge_{i<n} A_i \to B_n$

so $\{A_i\}_{i<n} \vdash B_n$, contradicting the choice of B_n.

4.3.2. Proof of the proposition.
Let $A = \{A_i\}_{i<\omega}$ be as in lemma 4.3.1., let

$$D_i \equiv A_i \vee \neg A_i$$

$$E_i \equiv (\bigwedge_{\substack{j\leq n \\ j\neq i}} D_j) \& A_i$$

and define $\alpha \equiv \{E_i\}_{i\leq n}$ (n+1 flas)
α is obviously $\exists$-mute. Let, on the other hand, $\beta \subset \alpha$ contain m flas, $m \leq n$.
Since the basic derivability properties of A, are invariant under permutations, we may assume, w.l.g., that

$$\beta \equiv \{E_i\}_{i<m\leq n} .$$

Then, since $E_1 \in \beta$ and $n > 1$

$$\beta \vdash A_n \vee \neg A_n .$$

If β is $\exists$-mute, then

$$A_1, \ldots, A_m, D_1, \ldots, D_n \vdash_{IA} A_n$$

and $$A_1, \ldots, A_m \vdash_{CA} A_n ,$$

contradicting the basic property of A.

5. <u>$\exists$-stable theories</u>.

5.0 It was conjectured by Łukasiewicz [52] that the "disjunction property" of a theory T

$$\vdash_T A \vee B \Longrightarrow \vdash_T A \text{ or } \vdash_T B$$

characterizes, among the propositional calculi, the intuitionistic prop. cal. from above; i.e.- no proper extension T of int. prop. cal. possess the "disjunction property".

This conjecture was refuted by a counter-example (Kreisel-Putnam [57]), and then - by an infinity of (finitely axiomatizable) counter-examples (Gabbay-de Jongh [69]).

Kleene [62] proved a stronger property of the int. prop. cal., namely -

$$C \vdash_T A \vee B \Longrightarrow C \vdash_T A \text{ or } C \vdash_T B$$

for exactly those C which are characterized by a given property (C|C). He subsequently conjectured that this property characterizes the int. prop. cal. from above, a conjecture which was proved by de Jongh [70].

An analogous property for intuitionistic arith. is:

(*) $$\alpha \vdash_{IA} \exists x Ax \Longrightarrow \exists n \quad \alpha \vdash_{IA} A\bar{n}$$

($\exists xAx$ closed) exactly when α is $\exists$-mute.

But for IA this property is <u>not</u> a characterization from above among theories of arithmetic. I.e. - if we define "α is $\exists$-mute" with IA replaced throughout by T, then (*) holds for proper extensions T of IA. For instance, let β be the set of instances of the scheme

$$\forall x \neg\neg Ax \to \neg\neg \forall x Ax \quad ,$$

which is not provable in IA (vid. Kleene [52] p. 511, th. 63 (iv)). Then "α is $\exists$-mute" with $IA+\beta$ replacing throughout IA in def. 1.1 means exactly "$\alpha \cup \beta$ is $\exists$-mute" (relative to IA now), and (*) for $IA+\beta$ follows.

We focus now our attention on extensions of IA which satisfy (*).

5.1. Definition: An extension T of IA is $\exists$-stable if whenever α is $\exists$-mute relative to T (i.e. - when IA is everywhere replaced by T in def. 1.1) then

$$\alpha \vdash_T \exists x Cx \implies \exists n \quad \alpha \vdash_T C\bar{n}$$

for every closed fla $\exists xCx$.

5.2. Existentially-valid inference rules.
Definition: A rule of inference ρ -

$$\frac{\{A_i\}_i}{B} \rho$$

is valid (modulo α and w.r.t. a theory T) if for each assignment * of numerals to parameters,

$$\{\alpha \vdash_T A_i^*\}_i \implies \alpha \vdash_T B^* \ .$$

We say that ρ is $\exists$-valid if it is valid modulo α for every $\exists$-mute set of flas α (and w.r.t. IA).

5.3. Examples: (1) The uniform reflection principle

$$\frac{\mathbb{P}[A]}{A} RP$$

where

$$\mathbb{P}[A] \equiv_{Df} \exists x\, Prov_{IA}\, (x, \ulcorner A \urcorner).$$

For suppose that * is an assignment of numerals to parameters, and that $\alpha \vdash \exists x\, Prov\, (x, \ulcorner A \urcorner)^*$, with α $\exists$-mute. Then

$$(+)\ \alpha \vdash Prov(\bar{n}, \ulcorner A \urcorner)^* \qquad \text{for some n.}$$

But $\quad Prov(\bar{n}, \ulcorner A \urcorner)^*$ is decidable in IA, so either

(i) $\vdash Prov(\bar{n}, \ulcorner A \urcorner)^*$, and then - since $Prov(\bar{n}, \ulcorner A \urcorner)^* \equiv Prov(\bar{n}, \ulcorner A^* \urcorner)$ - taking the derivation whose Gödel-number is n - we get a derivation for

$$\vdash_{IA} A^*$$

and ipso facto,

$$\alpha \vdash A^*;$$

or (ij) $\vdash \neg Prov(\bar{n}, \ulcorner A \urcorner)^*$, and then from (+) $\alpha \vdash 0 = 1$, and
$\alpha \vdash A$.

(2) Let $F = \{\exists x F_i x\}_i$ be a set of closed flas s.t. for every (classically-) true $(\exists xF) \in F$ there is exactly one number n_F s.t. $F\bar{n}_F$.
For instance -

$$F_0 = \{\exists x T(e,e,x)\}_e \text{ is such a set.}$$

Let ρ be the rule of instantiation for F, i.e. -

$$\frac{\exists x F x}{F\bar{n}_F}\ \rho \qquad\qquad ((\exists x F x) \in F)$$

where for false $(\exists xFx) \in F$ we define $n_F = 0$, say.

ρ is $\exists$-valid trivially. The rule of instantiation for the F_0 defined above is of course not recursively definable.

(3) We shall see in the sequel that Markov's scheme

$$\neg\forall x Px \rightarrow \exists x \neg Px \qquad \text{(P qu.-free)}$$

is not $\exists$-valid (though it is valid).

5.4. We understand, from now on, that when ρ is asserted to be an $\exists$-valid rule, then we are also given an assignment of derivations to derivations which attach to a derivation Δ for $\alpha \vdash A$ a derivation Δ^{ρ} for $\alpha \vdash B$, whenever $\frac{A}{B}\rho$ with closed A,B and $\exists$-mute α.

When ρ is an inference-rule, we write IA^{ρ} for IA extended with the rule ρ.

Theorem: If ρ is $\exists$-valid, then IA^{ρ} is $\exists$-stable.

Let us first stress some properties of derivations in IA^{ρ}.

5.4.1. Definition: Let ρ be valid modulo α. A ρ-α reduction is a transformation of derivations of the form

$$\begin{array}{c} \alpha \\ \Delta_i \\ \dfrac{A_i}{B}\,\rho \end{array} \longmapsto \begin{array}{c} \alpha \\ \Delta^{\rho} \\ B \end{array} \qquad (\Delta^{\rho}\ \rho\text{-free})$$

(provided all open assumption of the original derivation are in α, and A_i, B are closed). A derivation of IA^{ρ} is $\langle\rho,\alpha\rangle$-normal if it cannot be reduced, either by a ρ-α reduction, or by one of the reductions defined by Prawitz [70]. If ρ is $\exists$-valid, we say that a derivation is ρ-normal, if it is $\langle\rho,\alpha\rangle$-normal for every $\exists$-mute α. Let ρ be now some $\exists$-valid rule.

5.4.2. Lemma: Every derivation can be transformed by a finite number of reductions into a ρ-normal derivation (without redundant parameters).

5.4.3. Remark: The proof of the lemma is not an immediate corallary of the normalization theorem. If we normalize a derivation - except for ρ-reductions, and turn then to the ρ-reductions, then new logical detours may appear, even if Δ^{ρ} is always normal.

If, conversely, we first do with all ρ-reductions, and turn then to usual reductions, then new possible ρ-reductions may be created. That is, we may have an instance of →-reductions of the form

$$\begin{array}{c} \dfrac{\begin{matrix}\Delta \\ C\end{matrix} \qquad \dfrac{\begin{matrix}\overset{(1)}{[C]} \\ \Sigma_1 \\ \dfrac{A}{B}\,\rho \\ \Sigma_2 \\ D\end{matrix}}{C \to D}(1)}{D} \\ \Pi \end{array} \quad \underset{(\to\text{-reduction})}{\longmapsto} \quad \begin{array}{c} \Delta \\ [C] \\ \Sigma_1 \\ \dfrac{A}{B}\,\rho \\ \Sigma_2 \\ D \\ \Pi \end{array}$$

The indicated instance of ρ may be irreducible in the original derivation, because the open assumptions of $\begin{matrix}[C] \\ \Sigma_1\end{matrix}$ do not form an ∃-mute set of flas, while it is reducible in the reduct shown on the right, because the open assumptions of $\begin{matrix}\Delta \\ [C] \\ \Sigma_1\end{matrix}$ do form such a set.

Also, new ρ-detours may arrise from ∀-reductions, e.g. -

$$\begin{array}{c} \forall xAx \\ \dfrac{Aa}{Ba}\,\rho \\ \forall xBx \\ Bo \end{array} \quad \xrightarrow[(\forall\text{-reduction})]{} \quad \begin{array}{c} \forall xAx \\ \dfrac{Ao}{Bo}\,\rho \end{array}$$

If $\{\forall xAx\}$ is $\exists$-mute then the indicated instance of ρ is irreducible in the original derivation, but reducible in the $\forall$-reduced one.

5.4.4. Proof of the lemma

We just have to modify slightly Prawitz [70] proof of the normalization theorem for arithmetic. All notions are relativized to the extended system. Thus, to the definition of strong validity we add the clause:

$$\begin{array}{c} \alpha \\ \Delta \\ \dfrac{A}{B}\,\rho \end{array} \qquad \text{is } \rho\text{-strongly-valid if either}$$

(i) α is not $\exists$-mute and $\begin{bmatrix} \alpha \\ \Delta \\ A \end{bmatrix}$ is ρ-strongly-valid,

or (ij) α is $\exists$-mute, and for every assignment $*$ of numerals to parameters,

$$\begin{array}{c} \alpha^* \\ (\Delta^*)^\rho \\ A^* \end{array} \qquad \text{is } (\rho\text{-})\text{strongly-valid.}$$

The modifications required in the proof of the normalization-theorem are straightforward, and we leave them out.

5.4.5. Proof of the theorem.

The proof is completely analogous to that of lemma 1.2.2. and theorem 1.2. The only argument to be added is this: In the proof of lemma 1.2.2., no instance of ρ may occur in the branch σ in Δ^1, because Δ^1 is ρ-normal.

The same argument is used for case 2 in the proof of th. 1.2.

5.4.6. Remark: Theorem 5.4. was proved, for the uniform reflection principle, by Scarpellini [72].

5.5. Corollary: Markov's scheme

$$\frac{\neg\forall x Px}{\exists x \neg Px}\ (M) \qquad\qquad (P \text{ qu.-free})$$

is not ∃-valid.

Proof: Take $\alpha = \{\neg G\}$ where

$$G \equiv \forall x \neg Prov(x, \ulcorner G \urcorner).$$

α is ∃-mute relative to any theory, because no ∃ occur in α.

If (M) is ∃-valid, then - using th. 5.4. - $IA^{(M)}$ is ∃-stable, and we can conclude, like in 1.3.5, that $\vdash_{CA} G$, which is not the case.

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